MATH 20D: Differential Equations Spring 2023
Homework 5

Lecturer: Finn McGlade   UC San Diego

Remember to list the sources you used when completing the assignment.
Below *NSS* is used to reference the text *Fundamentals of Differential Equations (9th edition) by Nagle, Saff, Snider*

Question (1). *For each integer $n \geq 0$, let f_n denote the function*

$$f_n \colon [0,\infty) \to \mathbb{R}, \qquad f_n(t) = t^n.$$

(a) By analyzing the improper integral $\int_0^\infty e^{-st}dt$ for $s \in \mathbb{R}$, show that the Laplace transform $\mathcal{L}\{f_0\}$ is given by the function

$$\mathcal{L}\{f_0\} \colon (0,\infty) \to \mathbb{R}, \qquad \mathcal{L}\{f_0\}(s) = \frac{1}{s}.$$

(b) Using integration by parts, show that if $n \in \mathbb{Z}_{\geq 1}$ then

$$\mathcal{L}\{f_n\}(s) = \frac{n}{s} \cdot \mathcal{L}\{f_{n-1}\}(s), \qquad s > 0.$$

(c) By combining the results of part (a) and (b), conclude that $\mathcal{L}\{f_n\}$ is given by the function

$$\mathcal{L}\{f_n\} \colon (0,\infty) \to \mathbb{R}, \qquad \mathcal{L}\{f_n\}(s) = \frac{n!}{s^{n+1}}.$$

(d) Using the result of (c) together with the translation property of the Laplace transform, calculate $\mathcal{L}\{e^{at}t^n\}$ where $n \in \mathbb{Z}_{\geq 0}$ and a is constant.

Question (2). *Using the integral definition $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)dt$, calculate the Laplace transforms of the functions listed below.*

(a) $f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t & 1 \leq t < \infty. \end{cases}$ ***(b)*** $f(t) = \begin{cases} \sin(2t), & 0 \leq t < 3, \\ 0, & 3 \leq t < \infty. \end{cases}$ ***(c)*** $f(t) = \begin{cases} e^{2t}, & 0 \leq t < 3, \\ 1 & 3 \leq t < \infty. \end{cases}$

Question (3). *By making use of the table of Laplace transforms provided on page 3, determine the following Laplace transforms.*

(a) $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$, ***(b)*** $\mathcal{L}\{t^3 - te^t + e^{4t}\cos(t)\}$, ***(c)*** $\mathcal{L}\{e^{3t}\sin(6t) - t^3 + e^t\}$,

(d) $\mathcal{L}\{(1 + e^{-t})^2\}$, ***(e)*** $\mathcal{L}\{t^4e^{5t} - e^t\cos(\sqrt{7}t)\}$, ***(f)*** $\mathcal{L}\{te^{2t}\cos(5t)\}$,

(g) $\mathcal{L}\{\sin^2(2t)\}$, ***(h)*** $\mathcal{L}\{e^{7t}\sin^2(2t)\}$, ***(i)*** $\mathcal{L}\{t\sin(2t)\sin(5t)\}$.

Question (4)**.** *Let f be a piecewise continuous function defined on $[0,\infty)$ and suppose there exist constants $\alpha > 0$ and $K > 0$ such that*

$$|f(t)| \leq Ke^{\alpha t} \qquad \textit{for all } t \geq 0.$$

(a) By implementing the estimate

$$\left| \int_0^\infty e^{-st} f(t) ds \right| \leq \int_0^\infty |e^{-st} f(t)| ds,$$

show that $|\mathcal{L}\{f\}(s)| \leq K \cdot \int_0^\infty e^{-(s-\alpha)t} dt$ for all s. Conclude that $\lim_{s\to\infty} \mathcal{L}\{f\}(s) = 0$.

(b) Now suppose $\lim_{t\to 0^+}[f(t)/t]$ exists so that $f(t)/t$ is piecewise continuous on $[0,\infty)$. Using the Leibniz rule, show that

$$\frac{d}{ds}\left(\mathcal{L}\{t^{-1}f(t)\}(s)\right) = -\mathcal{L}\{f\}(s).$$

(c) By combining the results of parts (a) and (b), show that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f\}(u) du.$$

Use your result to compute the Laplace transform $\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}$.

Question (5)**.** *In each of the parts below, determine the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}(t)$.*

(a) $F(s) = \frac{2}{s^2+4}$, ***(b)*** $F(s) = \frac{4}{s^2+9}$, ***(c)*** $F(s) = \frac{3}{(2s+5)^3}$,

(d) $F(s) = \frac{1}{s^5}$, ***(e)*** $F(s) = \frac{s-1}{2s^2+s+6}$, ***(f)*** $F(s) = \frac{6s^2-13s+2}{s(s-1)(s-6)}$,

(g) $F(s) = \frac{s+11}{(s-1)(s+3)}$, ***(h)*** $F(s) = \frac{7s^2-41s+84}{(s-1)(s^2-4s+13)}$, ***(i)*** $F(s) = \frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$,

(h) $s^2F(s) + sF(s) - 6F(s) = \frac{s^2+4}{s^2+s}$, ***(i)*** $sF(s) - F(s) = \frac{2s+5}{s^2+2s+1}$.

Question (6)**.** *The identity $\frac{d^n}{ds^n}\mathcal{L}\{f\}(s) = (-1)^n \mathcal{L}\{t^n f(t)\}(s)$ implies that*

$$\mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}\mathcal{L}\{f\}(s)\right\}(t) = (-t)^n f(t). \tag{0.1}$$

Use (0.1) to calculate the inverse Laplace transforms below

(a) $\mathcal{L}^{-1}\left\{\log\left(\frac{s+2}{s-5}\right)\right\}(t)$, ***(b)*** $\mathcal{L}^{-1}\left\{\log\left(\frac{s-4}{s-3}\right)\right\}(t)$,

(c) $\mathcal{L}^{-1}\left\{\log\left(\frac{s^2+9}{s^2+1}\right)\right\}(t)$, ***(b)*** $\mathcal{L}^{-1}\{\arctan(1/s)\}(t)$

A TABLE OF LAPLACE TRANSFORMS

	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1.	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
2.	$e^{at}f(t)$	$F(s-a)$
3.	$f'(t)$	$sF(s) - f(0)$
4.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
5.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
6.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)du$
7.	$\int_0^t f(v)dv$	$\frac{F(s)}{s}$
8.	$(f * g)(t)$	$F(s)G(s)$
9.	$f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$
10.	$f(t-a)u(t-a),\ a \geq 0$	$e^{-as}F(s)$
11.	$g(t)u(t-a),\quad a \geq 0$	$e^{-as}\mathcal{L}\{g(t+a)\}(s)$
12.	$u(t-a),\quad a \geq 0$	$\frac{e^{-as}}{s}$
13.	$\Pi_{a,b}(t),\quad 0 < a < b$	$\frac{e^{-sa} - e^{-sb}}{s}$
14.	$\delta(t-a),\quad a \geq 0$	e^{-as}
15.	e^{at}	$\frac{1}{s-a}$
16.	$t^n,\quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
17.	$e^{at}t^n,\quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
18.	$e^{at} - e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$
19.	$ae^{at} - be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$
20.	$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
21.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
22.	$t^{n-(1/2)},\quad n = 1, 2, \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
23.	$t^r,\quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
24.	$\sin bt$	$\frac{b}{s^2+b^2}$
25.	$\cos bt$	$\frac{s}{s^2+b^2}$
26.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
27.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
28.	$\sinh bt$	$\frac{b}{s^2-b^2}$
29.	$\cosh bt$	$\frac{s}{s^2-b^2}$
30.	$\sin bt - bt\cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$
31.	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
32.	$\sin bt + bt\cos bt$	$\frac{2bs^2}{(s^2+b^2)^2}$
33.	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
34.	$\sin bt\cosh bt - \cos bt\sinh bt$	$\frac{4b^3}{s^4+4b^4}$
35.	$\sin bt\sinh bt$	$\frac{2b^2s}{s^4+4b^4}$
36.	$\sinh bt - \sin bt$	$\frac{2b^3}{s^4-b^4}$
37.	$\cosh bt - \cos bt$	$\frac{2b^2s}{s^4-b^4}$
38.	$J_\nu(bt),\ \nu > -1$	$\frac{(\sqrt{s^2+b^2}-s)^\nu}{b^\nu\sqrt{s^2+b^2}}$